DESIGN AND MODELING OF INTERNALLY PRESSURIZED THICK-WALLED CYLINDER

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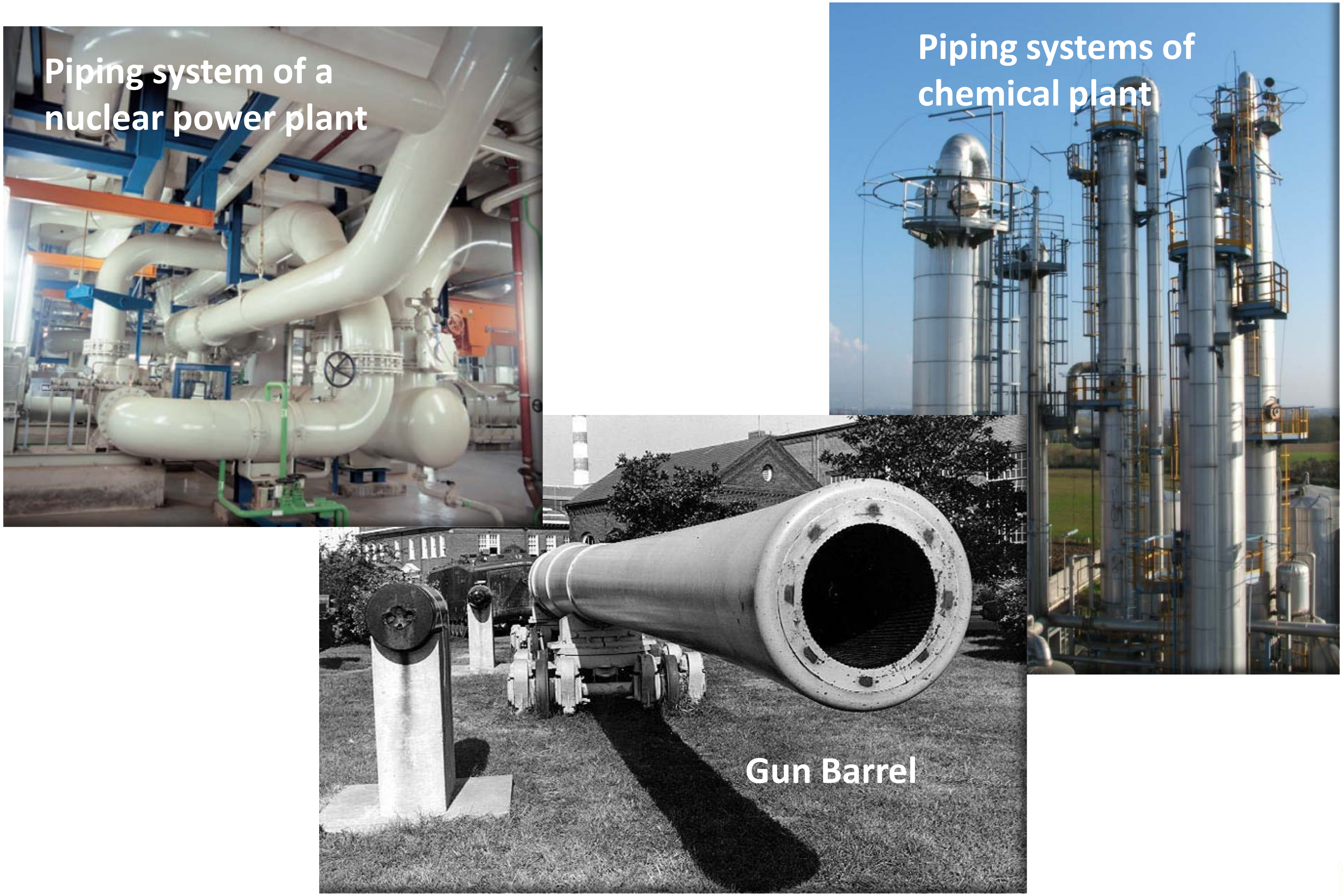
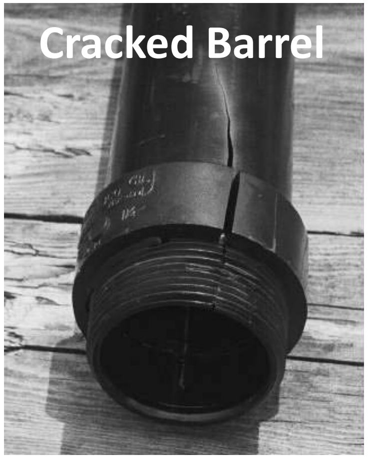
OUTLINE

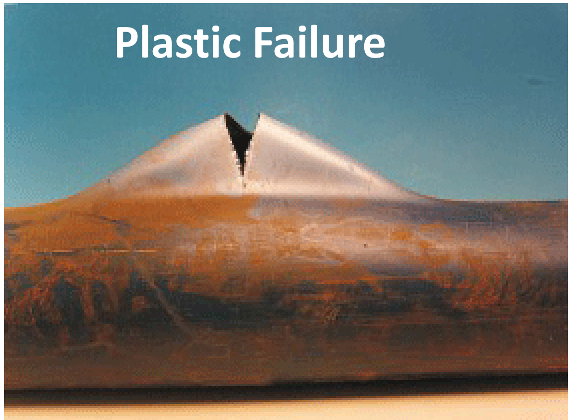
1. Introduction
2. Basic Concepts OF A Pressurized Thick-Walled Cylinder
3. Stress Analysis of A Single-Layer Pressurized Thick-Walled Cylinder
4. Stress Analysis of A Double-Layer Pressurized Thick-Walled Cylinder
5. Stress Analysis of A Composite-Wrapped Pressurized Thick-Walled Cylinder
6. Conclusions
7. Acknowledgements

1. INTRODUCTION



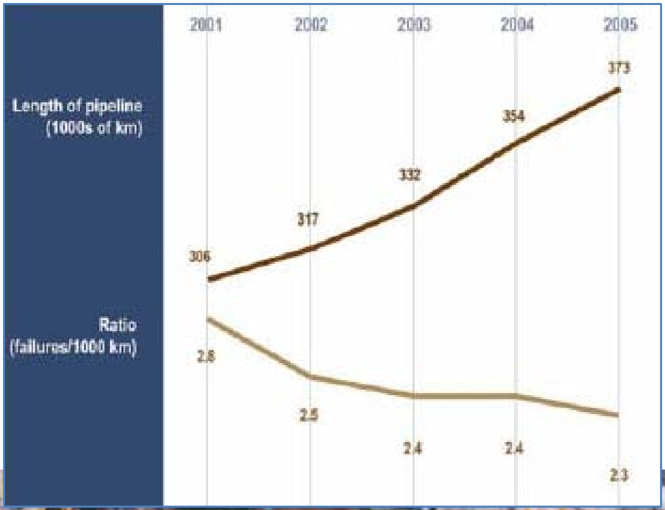
1. INTRODUCTION – CONT.

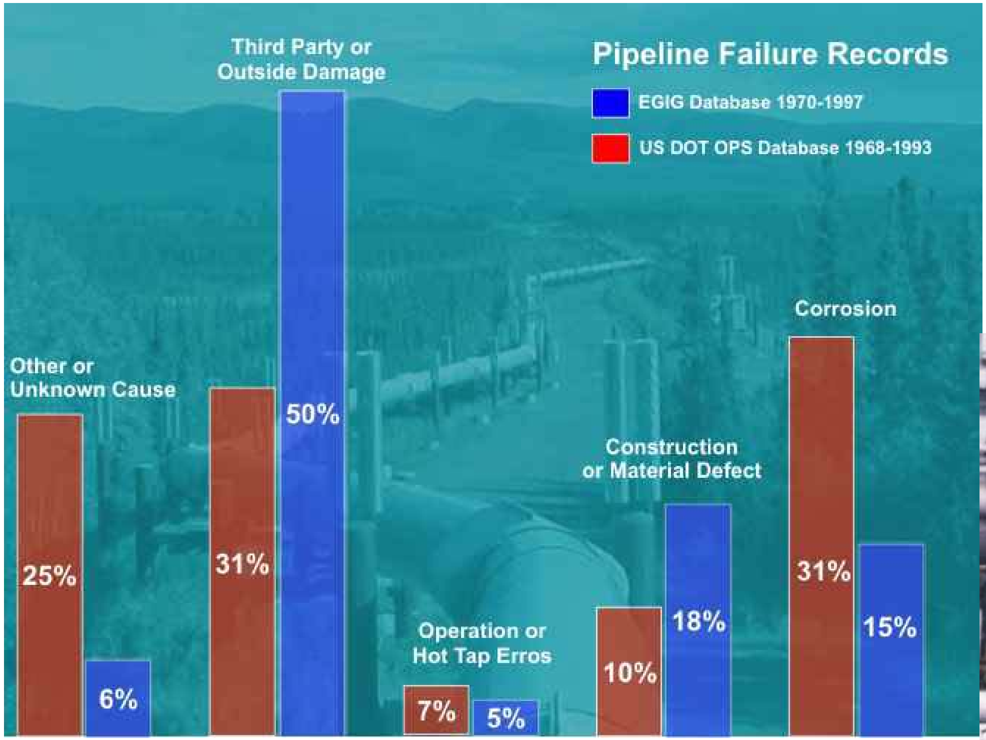

Cracked Barrel


Plastic Failure

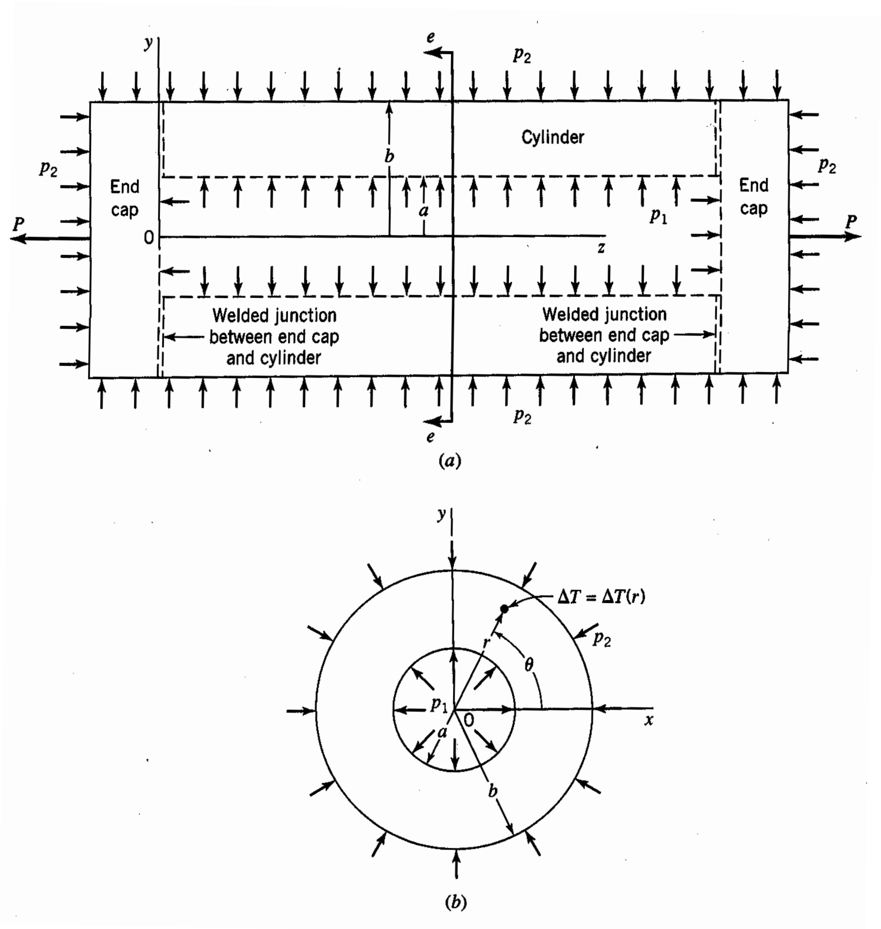

Oil & Gas Pipeline Failure

1. INTRODUCTION – CONT.



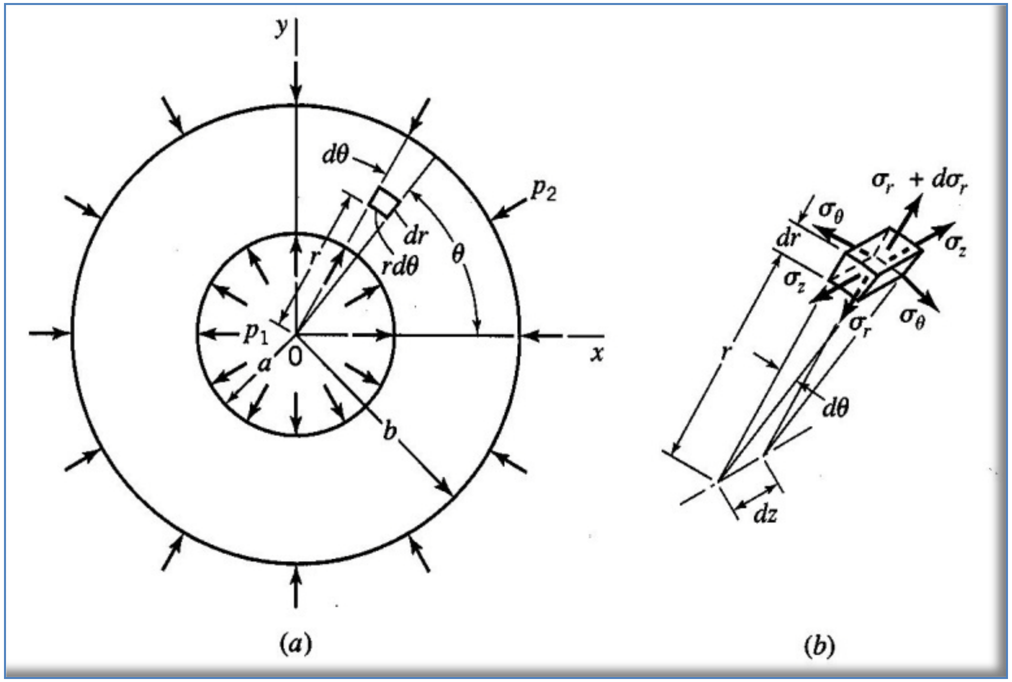


2. BASIC CONCEPTS OF A PRESSURIZED THICK-WALLED CYLINDER



Closed cylinder with internal pressure, external pressure, and axial loads. (*a*) Closed cylinder. (*b*) Section *e-e*.

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER



Stresses in thick-wall cylinder. (*a*) Thin annulus of thickness dz. (*b*) Cylindrical volume element of thickness dz.

Basic Assumptions:

(1). Static loads
(2). Isotropic and homogenous material
(3). Constant temperature
(4). Elasto-plastic and small deformation
(5). Ignoring axial load (stress)
(6). Cross section keeping plane after deformation

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Elastic Analysis:

Equilibrium Equation

$$\sigma_\theta = \sigma_r + r\frac{d\sigma_r}{dr} = \frac{d}{dr}(r\sigma_r)$$

Strain-Displacement Relations

$$\epsilon_r = \frac{\partial u}{\partial r}, \epsilon_\theta = \frac{u}{r}, \epsilon_z = \frac{\partial w}{\partial r} = constant$$

Strain Compatibility Condition

$$\epsilon_r = \epsilon_\theta + r\frac{d\epsilon_r}{dr} = \frac{d}{dr}(r\epsilon_\theta)$$

Hooke's Law (stress-strain relations)

$$\epsilon_r = \frac{1}{E}[\sigma_r - \upsilon\sigma_\theta]$$

$$\epsilon_\theta = \frac{1}{E}[\sigma_\theta - \upsilon\sigma_r]$$

$$\epsilon_z = \frac{1}{E}[-\upsilon(\sigma_r + \sigma_\theta)]$$

Stress Components under Internal and External Pressure

$$\sigma_r = \frac{p_1a^2 - p_2b^2 + (p_2 - p_1)(ab/r)^2}{b^2 - a^2}$$

$$\sigma_\theta = \frac{p_1a^2 - p_2b^2 - (p_2 - p_1)(ab/r)^2}{b^2 - a^2}$$

$$\sigma_r + \sigma_\theta = 2\frac{p_1a^2 - p_2b^2}{b^2 - a^2} = constant$$

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Stress Components under Internal Pressure Only

$$\sigma_r' = \frac{p_1 a^2(1-b^2/r^2)}{b^2-a^2}$$

$$\sigma_\theta' = \frac{p_1 a^2(1+b^2/r^2)}{b^2-a^2}$$

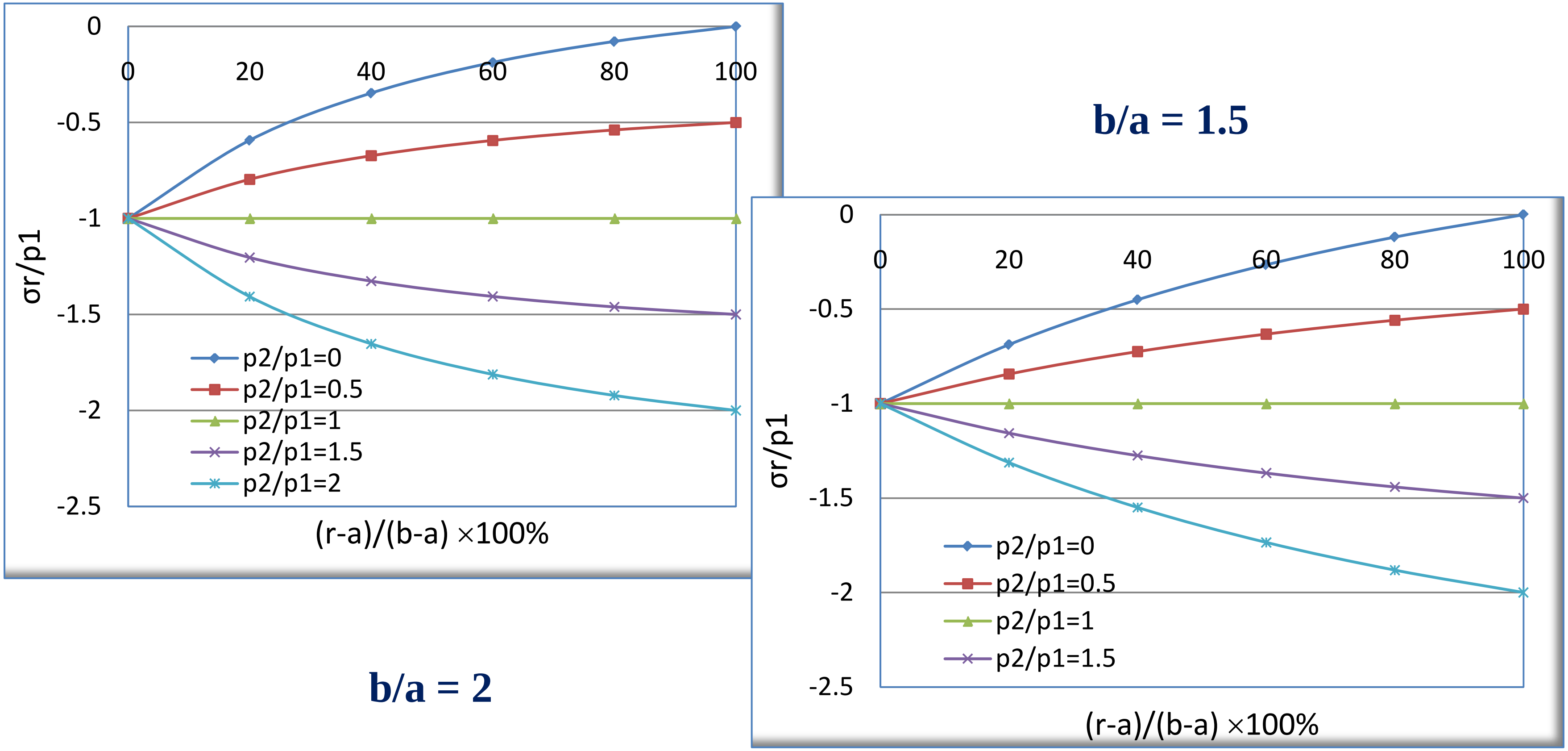
$$\sigma_r'' = \frac{p_2 b^2(a^2/r^2-1)}{b^2-a^2}$$

$$\sigma_\theta'' = -\frac{p_2 b^2(a^2/r^2+1)}{b^2-a^2}$$

Radial Displacement under Internal and External Pressure

$$u = \frac{r}{E(b^2-a^2)}\left[(1-2v)(p_1a^2-p_2b^2)+\frac{(1+v)a^2b^2}{r^2}(p_1-p_2)\right] \qquad (10)$$

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

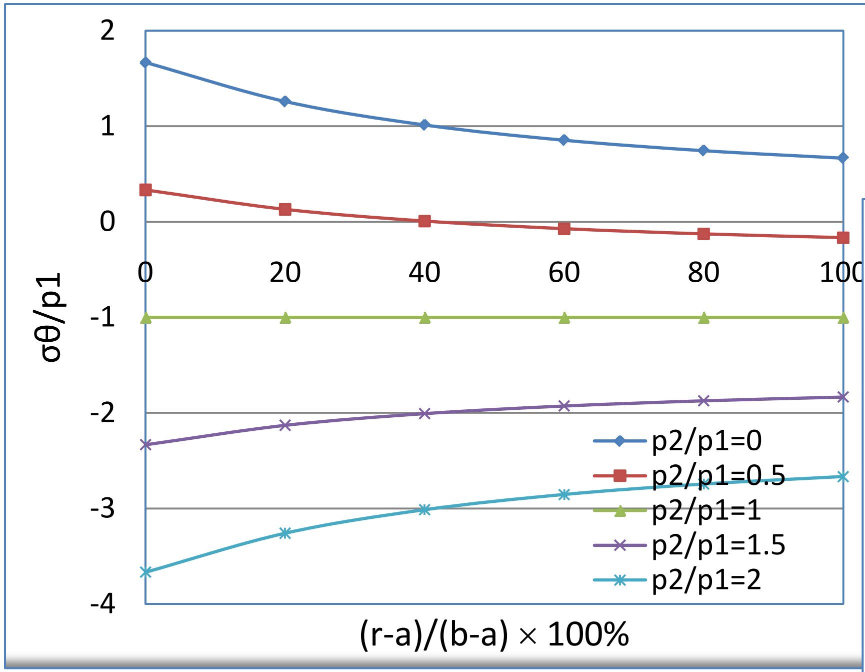


b/a = 1.5

b/a = 2

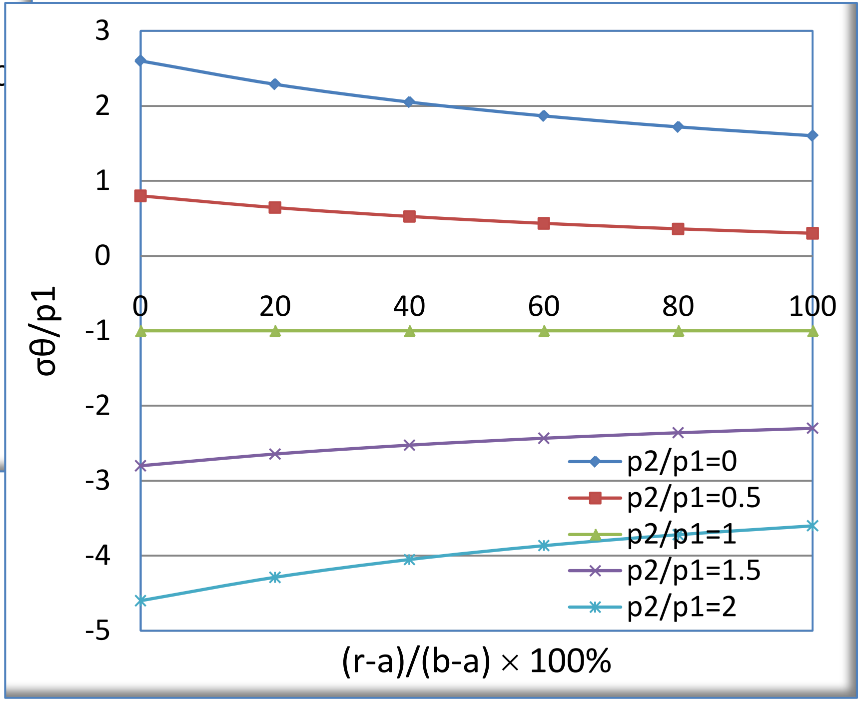
Radial Stress, σ_r , Distribution in A Single Layer Thick-Wall Cylinder.

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.



b/a = 2

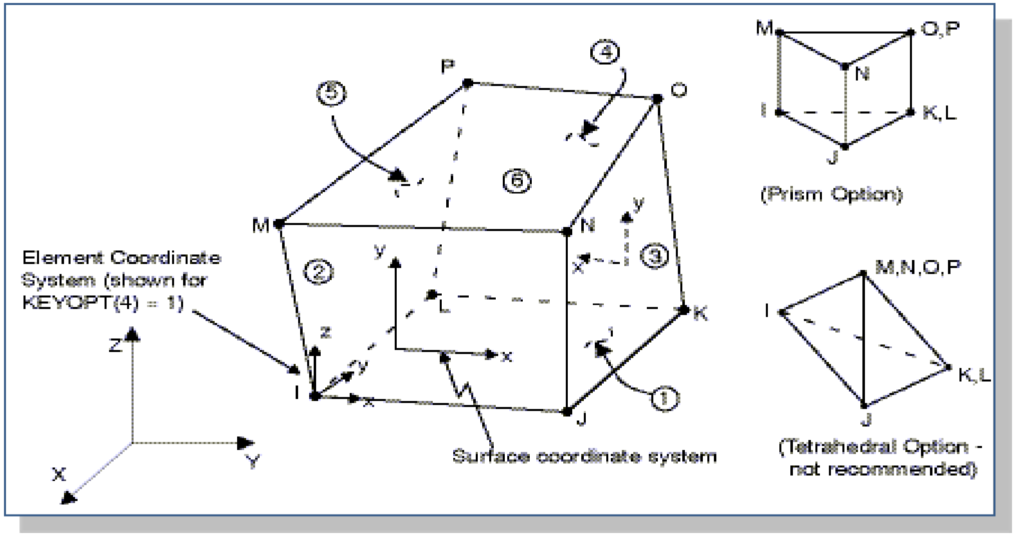
b/a = 1.5



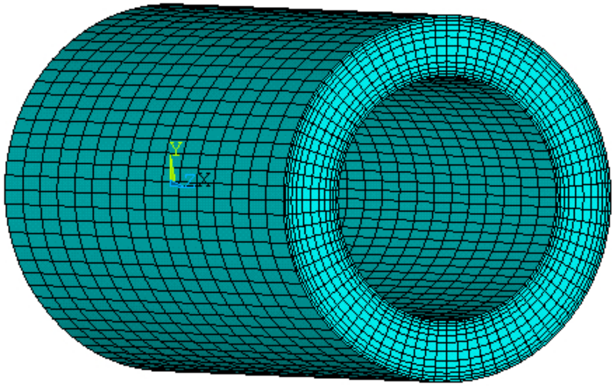
Hoop Stress, σ_θ, Distribution in A Single Layer Thick-Wall Cylinder.

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

3-D structural solid element

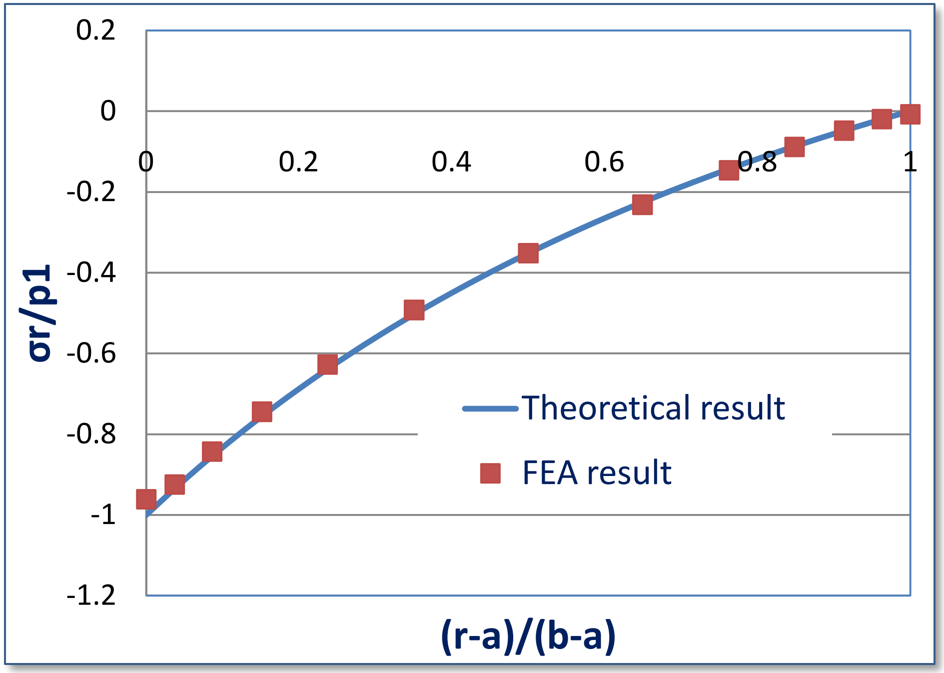


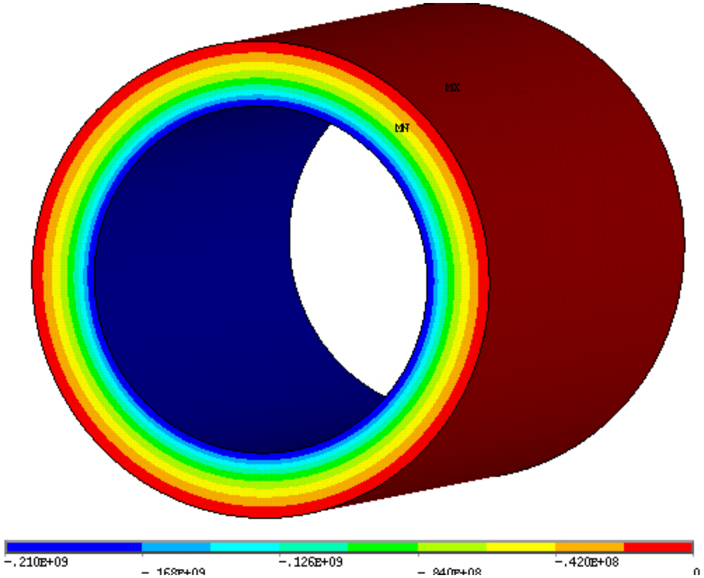
Finite Element Model



3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Comparison of the analytical results with FEA results of an internally pressurized thick wall cylinder.

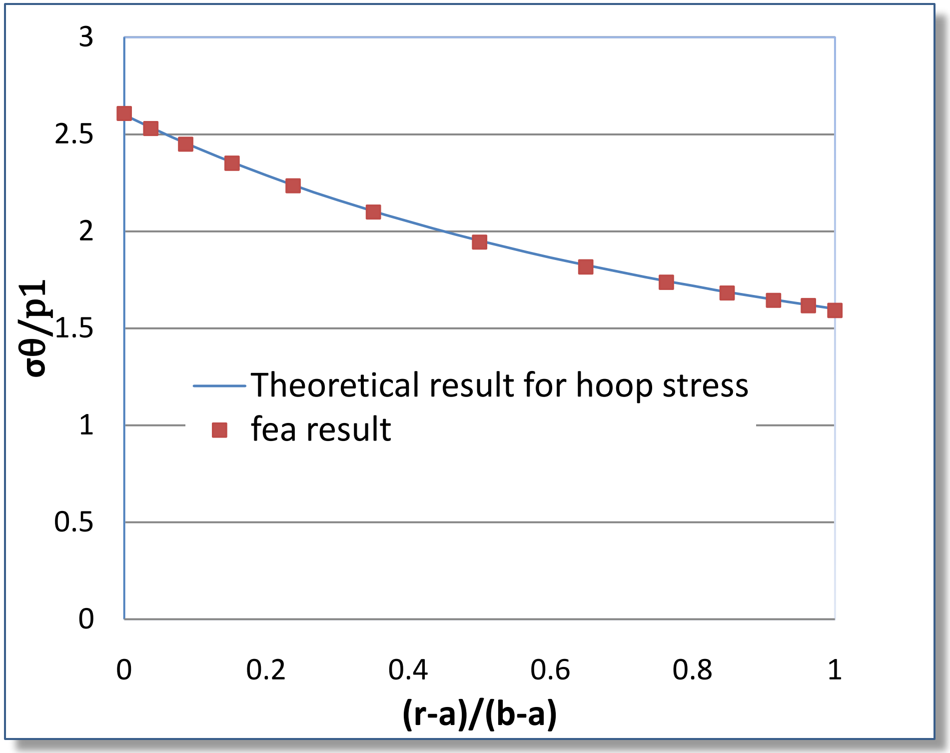


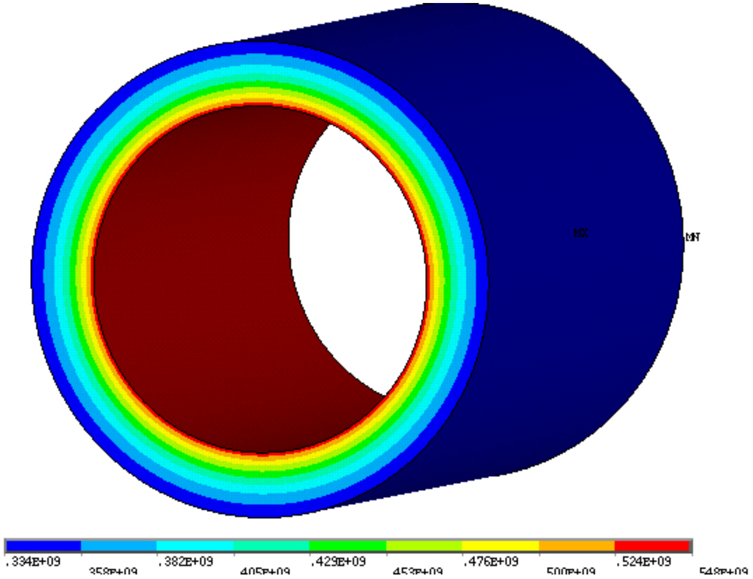


Radial Stress

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Comparison of the analytical results with FEA results of an internally pressurized thick wall cylinder.

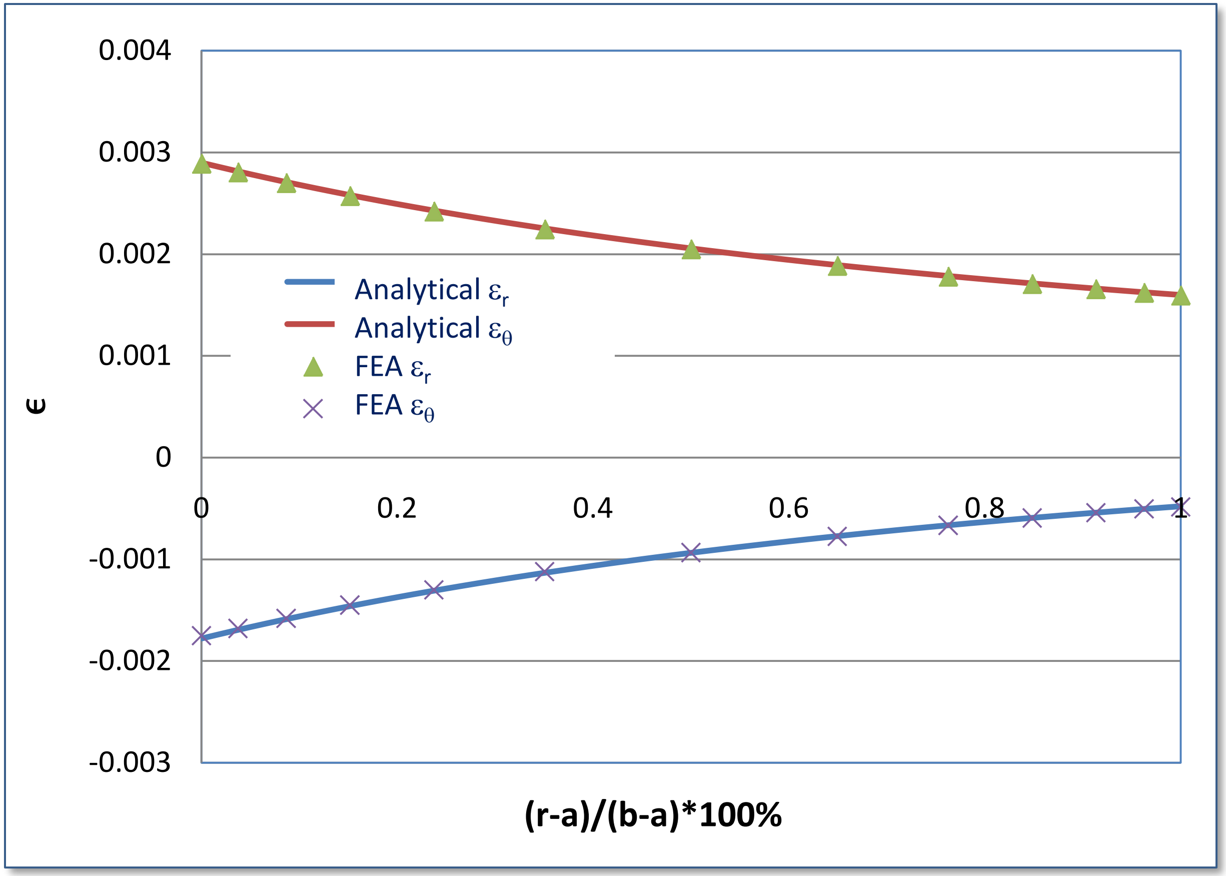




Hoop Stress

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Comparison of the analytical results with FEA results of elastic strains in an internally pressurized thick wall cylinder.



3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Elasto-Plastic Analysis of Autofrettage

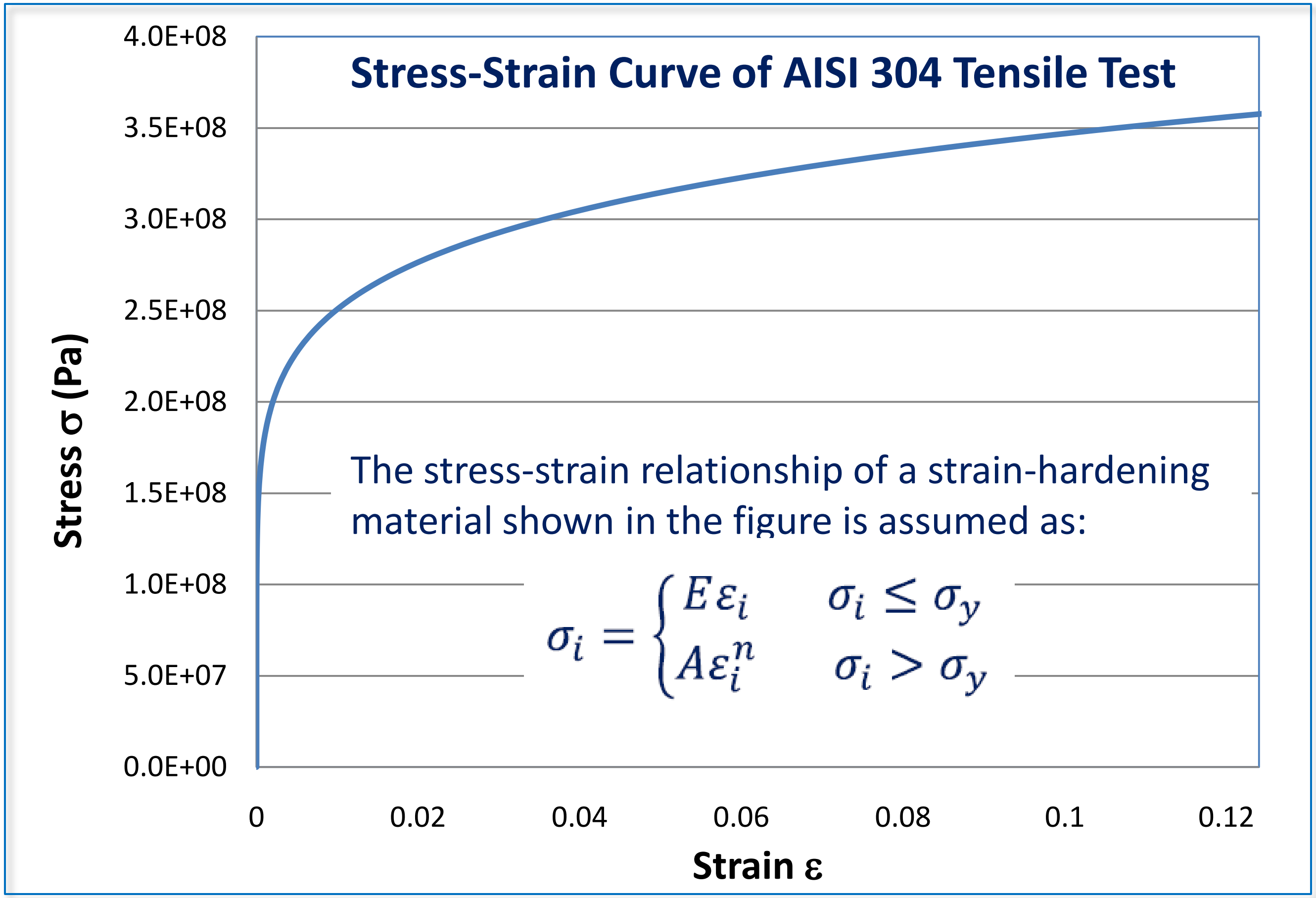
From Wikipedia:
Autofrettage is a metal fabrication technique in which a pressure vessel is subjected to enormous pressure, causing internal portions of the part to yield and resulting in internal compressive residual stresses. The goal of autofrettage is to increase durability of the final product. The technique is commonly used in manufacturing high-pressure pump cylinders, battleship and tank cannon barrels, and fuel injection systems for diesel engines. While some work hardening will occur, that is not the primary mechanism of strengthening.

When autofrettage is used for strengthening cannon barrels, the barrel is prebored to a slightly undersized inside diameter, and then a slightly oversized die is pushed through the barrel. The amount of initial underbore and size of the die are calculated to strain the material past its elastic limit into plastic deformation, sufficiently far that the final strained diameter is the final desired bore.

The technique has been applied to the expansion of tubular components down hole in oil and gas wells. The method has been patented by the Norwegian oil service company, READ, which uses it to connect concentric tubular components with sealing and strength properties outlined above.

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Elasto-Plastic Analysis of Autofrettage


Stress-Strain Curve of AISI 304 Tensile Test

The stress-strain relationship of a strain-hardening material shown in the figure is assumed as:

$$\sigma_i = \begin{cases} E\varepsilon_i & \sigma_i \le \sigma_y \\ A\varepsilon_i^n & \sigma_i > \sigma_y \end{cases}$$

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Elasto-Plastic Analysis

It can be assumed that the elastic zone of vessel is a cylinder of inner radius ρ and outer radius *b* which is subjected to internal pressure p_ρ

$$\sigma_\theta = \frac{p_\rho \rho^2}{b^2 - \rho^2}\left(1 + \frac{b^2}{r^2}\right), \quad \sigma_r = \frac{p_\rho \rho^2}{b^2 - \rho^2}\left(1 - \frac{b^2}{r^2}\right)$$

Where p_ρ in plane-strain and plane-stress conditions is obtained as:

$$p_\rho = \frac{\sigma_y}{\sqrt{3}}\left(1 - \frac{\rho^2}{b^2}\right), \quad p_\rho = \frac{\sigma_y (b^2 - \rho^2)}{\sqrt{(3b^4 + \rho^4}}$$

The elastic-limit pressure p_e and the plastic-limit pressure p_y are:

$$p_e = \frac{\sigma_y}{\sqrt{3}}\left(1 - \frac{a^2}{b^2}\right), \quad p_y = \frac{\sigma_y}{\sqrt{3}}\frac{1}{n}\left(\frac{b^{2n}}{a^{2n}} - 1\right)$$

The relation between internal pressure and the radius of the elastic-plastic boundary in plane-strain and plane-stress condition is determined as:

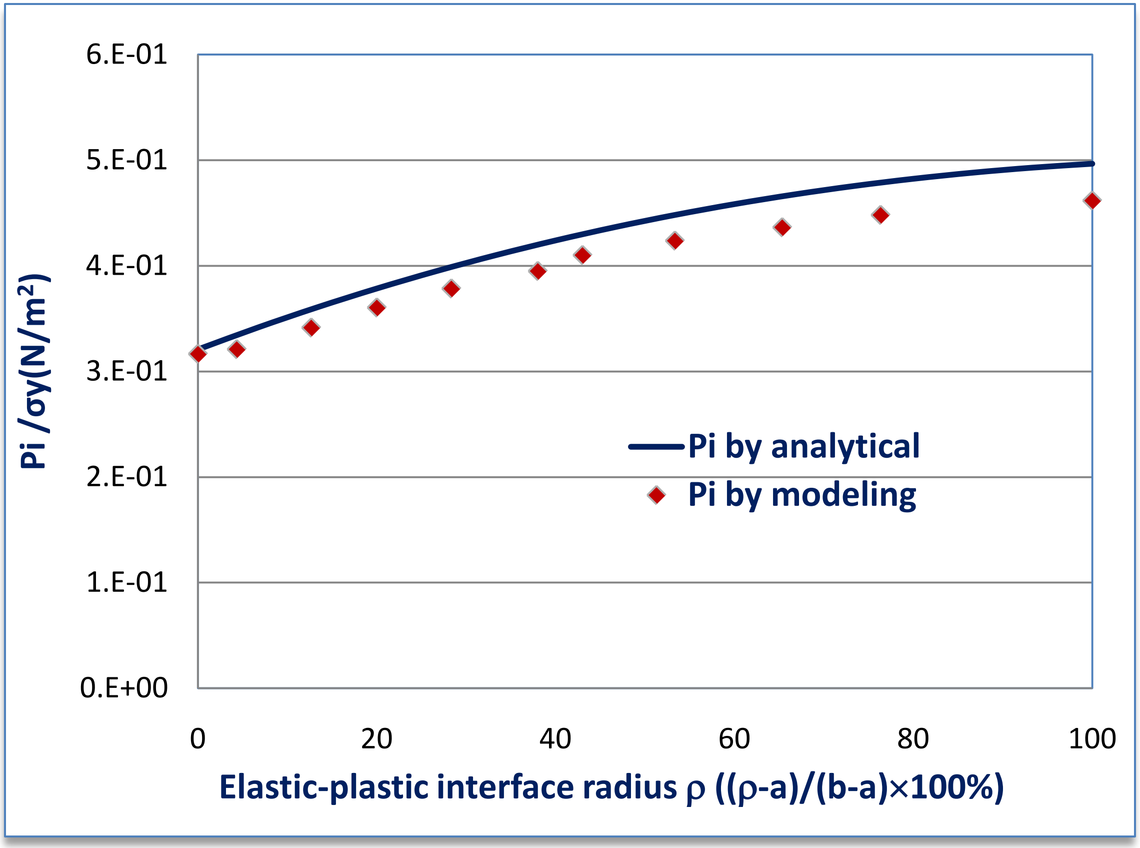
$$p_i = \frac{\sigma_y}{\sqrt{3}}\left[\left(1 - \frac{\rho^2}{b^2}\right) + \frac{1}{n}\left(\frac{\rho^{2n}}{a^{2n}} - 1\right)\right],$$

$$\begin{cases} \dfrac{p_i}{\sigma_y} = \dfrac{2}{\sqrt{3}}\left|\dfrac{\cos(\phi_\rho + \phi_n)}{\cos(\phi_a + \phi_n)}\right|^{(3n^2+n)/(3n^2+1)} \\ \qquad \times \exp\left[\dfrac{\sqrt{3}n(n-1)}{3n^2+1}(\phi_a - \phi_\rho)\right] \cos\phi_a, \\ \rho = a\sqrt{\dfrac{\sin(\phi_a + \pi/6)}{\sin(\phi_\rho + \pi/6)}}\left|\dfrac{\cos(\phi_\rho + \phi_n)}{\cos(\phi_a + \phi_n)}\right|^{2n/(3n^2+1)} \\ \qquad \times \exp\left[\dfrac{\sqrt{3}}{2}\dfrac{1-n^2}{3n^2+1}(\phi_\rho - \phi_a)\right], \\ \phi_\rho = \cos^{-1}\dfrac{\sqrt{3}}{2}\dfrac{(b^2 - \rho^2)}{\sqrt{(3b^4 + \rho^4)}}, \end{cases}$$

where ϕ_n is defined as $\cos^{-1}(\sqrt{3}n/\sqrt{3n^2+1})$

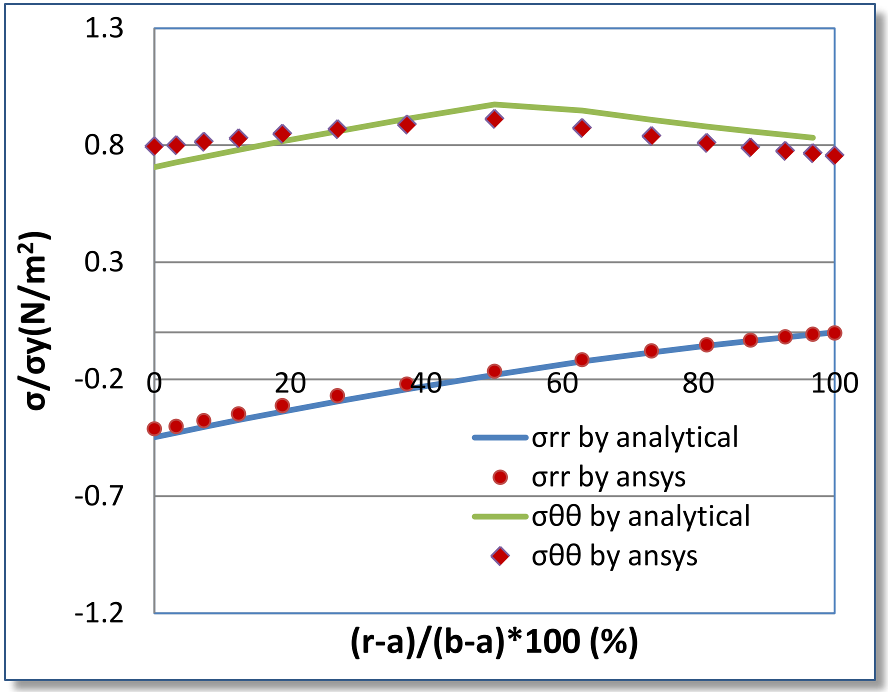
3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

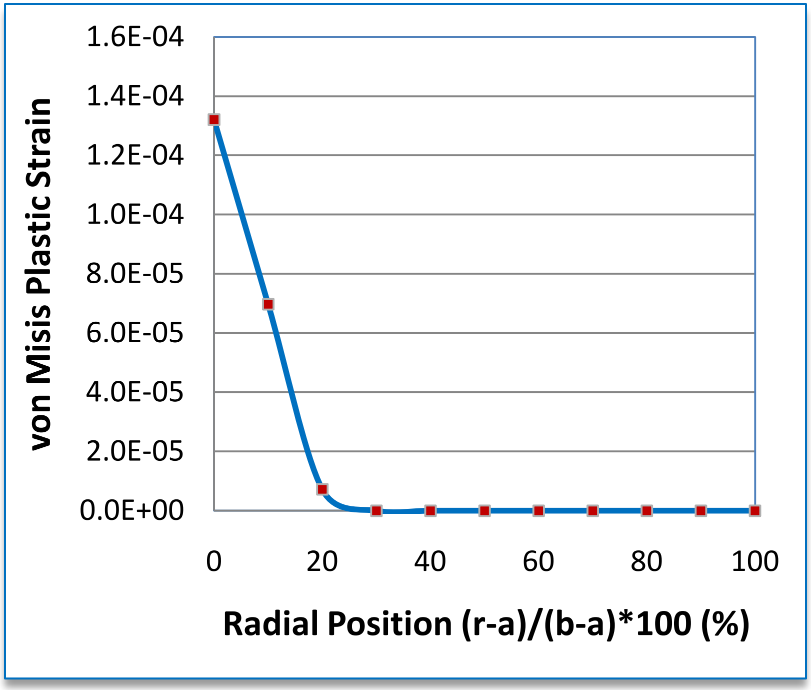
Comparison of the analytical results with FEA results of the elastic-plastic interface radius vs. the internal pressure.



3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Comparison of the analytical results with FEA results of the strain and stress during pressuring process



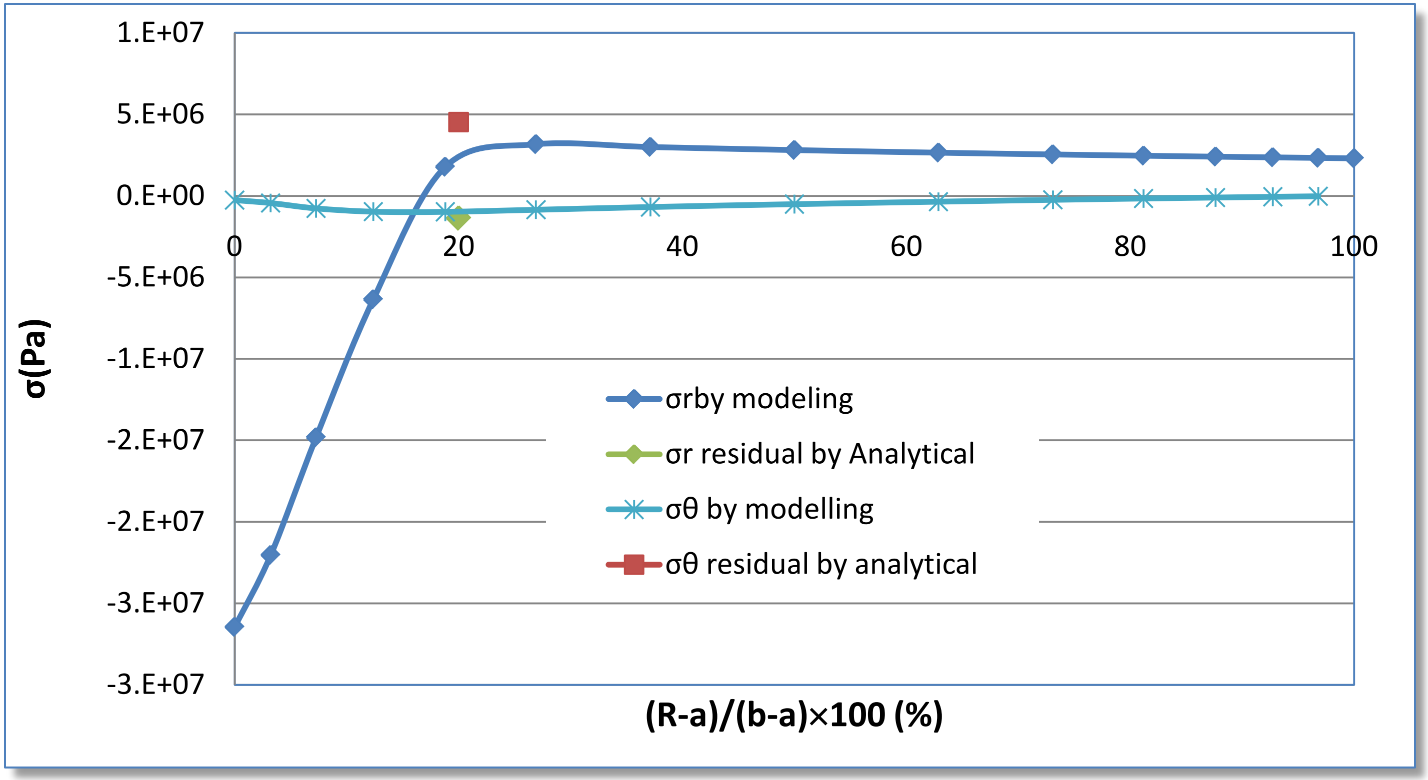


Stresses at pi=86.9 MPa and ρ=53.33%

Strain at pi=65.8 MPa and ρ=21.15%

3. STRESS ANALYSIS OF A SINGLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

FEA results of residual stresses of autofrettaged cylinder



Residual stresses of autofrettaged cylinder (pi=7.39 MPa and ρ=20.0%

4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER

Elastic Analysis:

During Assembling, the radial displacement of the inner layer(p_1=0):

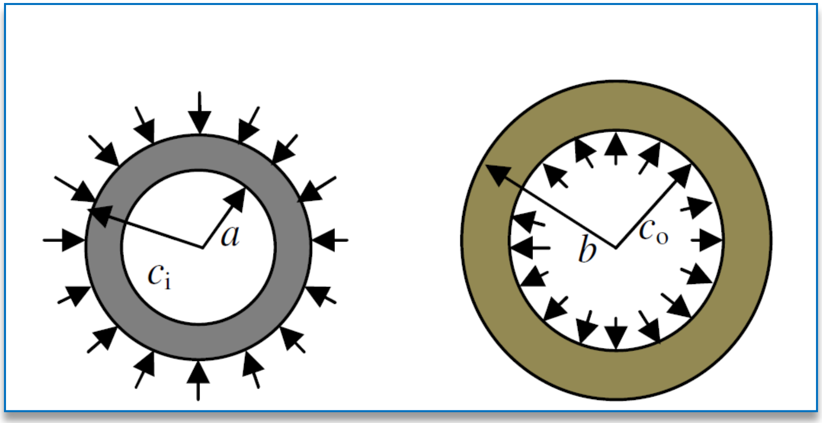
$$u_i' = \frac{-r p_i c_i^2}{E(c_i^2-a^2)}\left[(1-2v)+\frac{(1+v)a^2}{r^2}\right]$$

During assembling, the radial displacement of the outer layer (p_2=0):

$$u_o' = \frac{r p_i c_o^2}{E(b^2-c_o^2)}\left[(1-2v)+\frac{(1+v)b^2}{r^2}\right]$$

$$u_i' + c_i = u_o' + c_o = c$$

Initial geometric condition of the composite cylinders before assembly ($\Delta = c_i - c_o$).



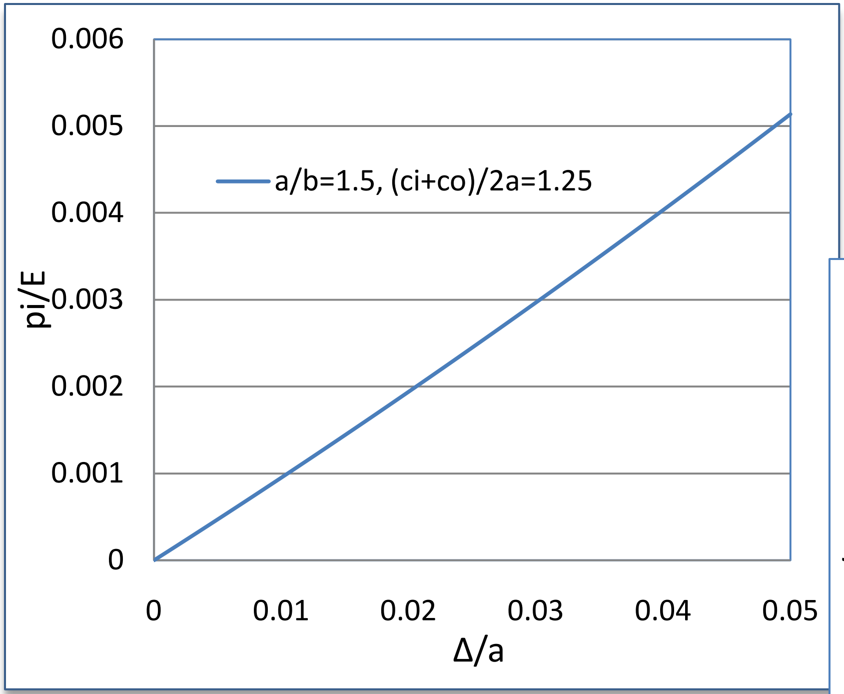
During pressuring ($p_1 \neq 0$) after assembling, the radial displacement of the outer and inner layer are:

$$u_o = \frac{r}{E(b^2-c_o^2)}\left[(1-2v)(p_i c_o^2 - p_2 b^2)+\frac{(1+v)c_o^2 b^2}{r^2}(p_i-p_2)\right]$$

$$u_i = \frac{r}{E(c_i^2-a^2)}\left[(1-2v)(p_1 a^2 - p_i c_i^2)+\frac{(1+v)a^2 c_i^2}{r^2}(p_1-p_i)\right]$$

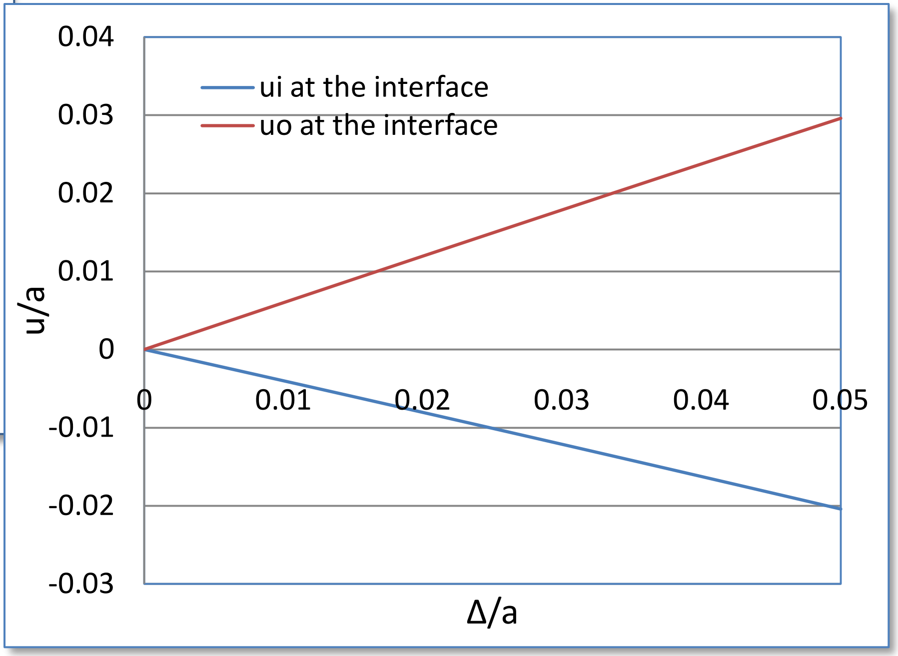
$$\Delta = \frac{c_o p_i}{E(b^2-c_o^2)}[(1-2v)c_o^2+(1+v)b^2] + \frac{c_i p_i}{E(c_i^2-a^2)}[(1-2v)c_i^2+(1+v)a^2]$$

4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

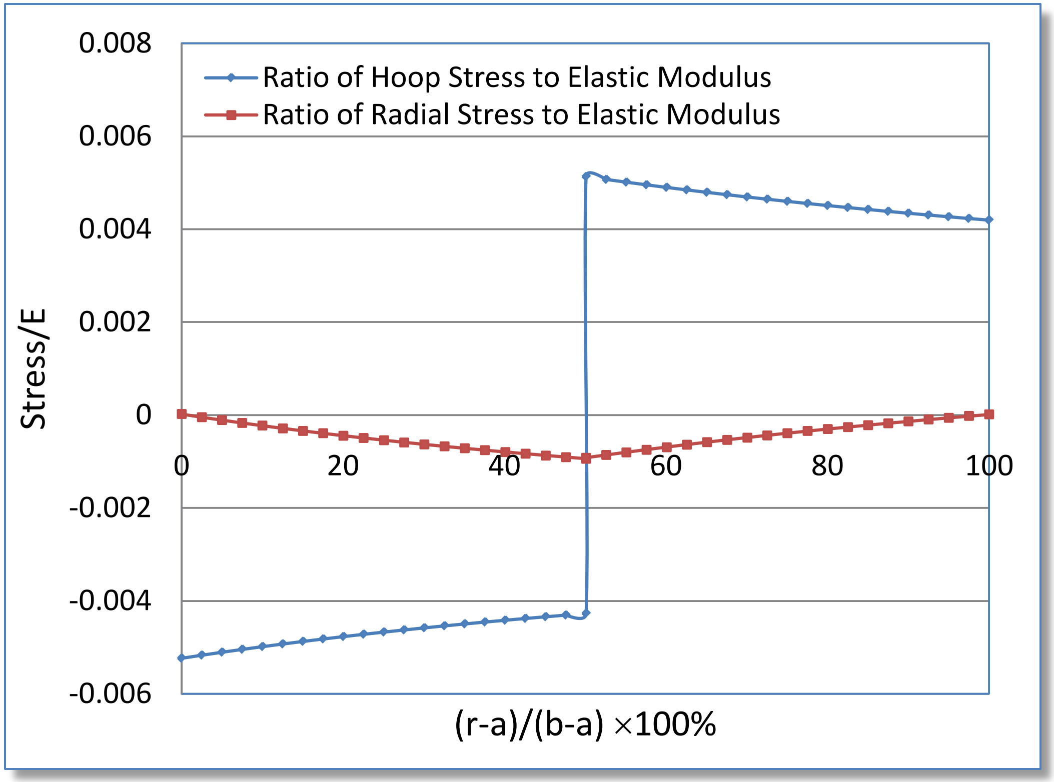


Relationship between the difference of Δ/a and p_i/E (b/a=1.5 and (c_i+c_o)/2a=1.25, and υ=0.3).

Relation between Δ and *u*.

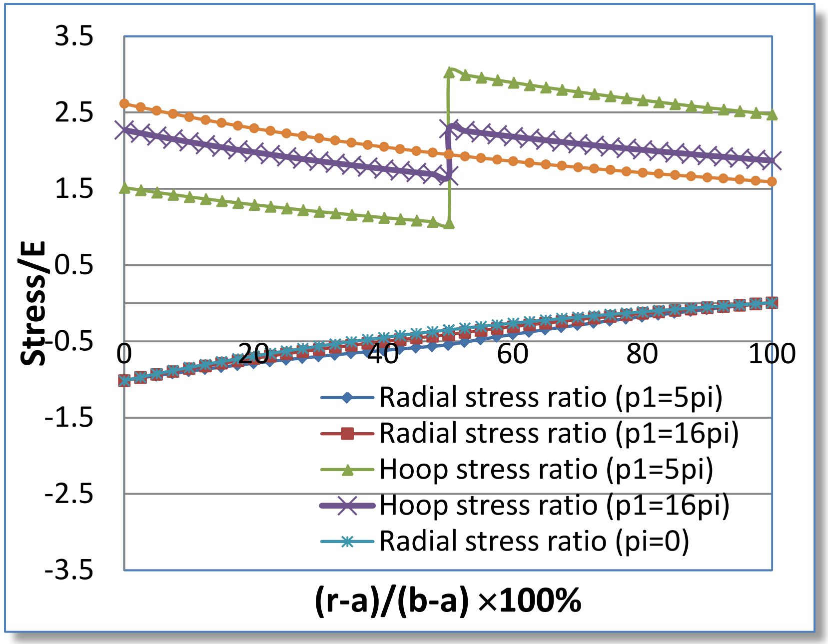


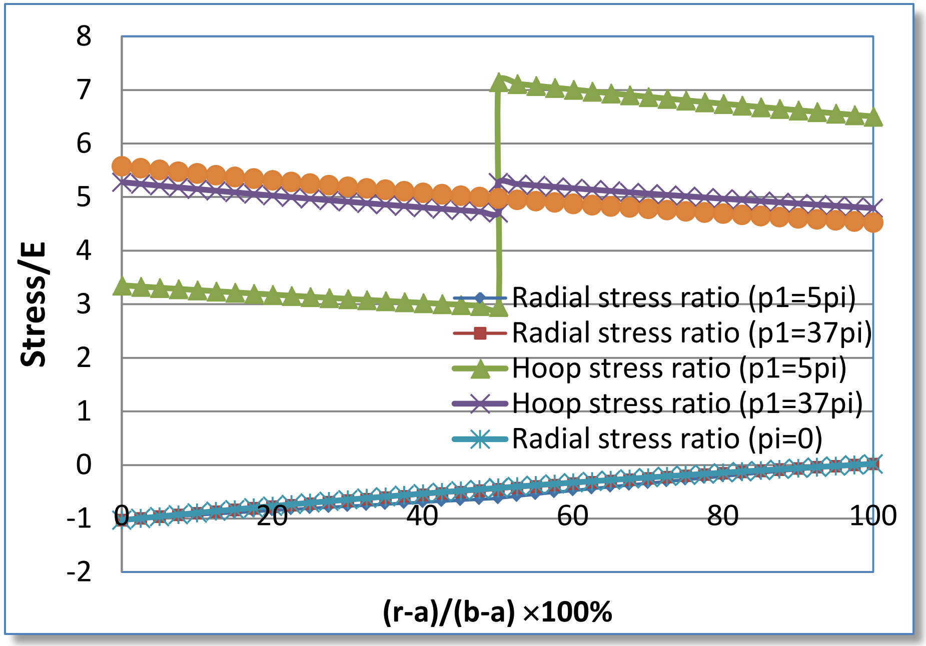
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.



The radial and hoop stress distributions by prestressed assembly (b/a=1.5 and $(c_i+c_o)/2a$=1.25, and υ=0.3).

4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.





The radial and hoop stress distributions by internal pressure p_1 and the assembly pressure p_i (b/a=1.5 and $(c_i+c_o)/2a$=1.25, and υ=0.3).

The radial and hoop stress distributions by internal pressure p_1 and the assembly pressure p_i (b/a=1.2 and $(c_i+c_o)/2a$=1.1, and υ=0.3).

4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Coulomb friction model, two contacting surfaces can carry shear stresses up to a certain magnitude across their interface before they start sliding relative to each other. The state is known as sticking. The Coulomb friction model is defined as:

$$\tau_{\lim} = \mu P + b \text{ and } |\tau| \le \tau_{\lim}$$

where:

$\tau_{\lim}$ = limit shear stress;

τ = equivalent shear stress;

μ = frictional coefficient;

P = contact normal pressure;

b = contact cohesion, providing sliding resistance even with zero normal pressure.

An exponential friction model can be used to smooth the transition between the static coefficient of friction and the dynamic coefficient of friction

$$\mu(\upsilon) = \mu_d + (\mu_s - \mu_d)e^{-c|\upsilon|}$$

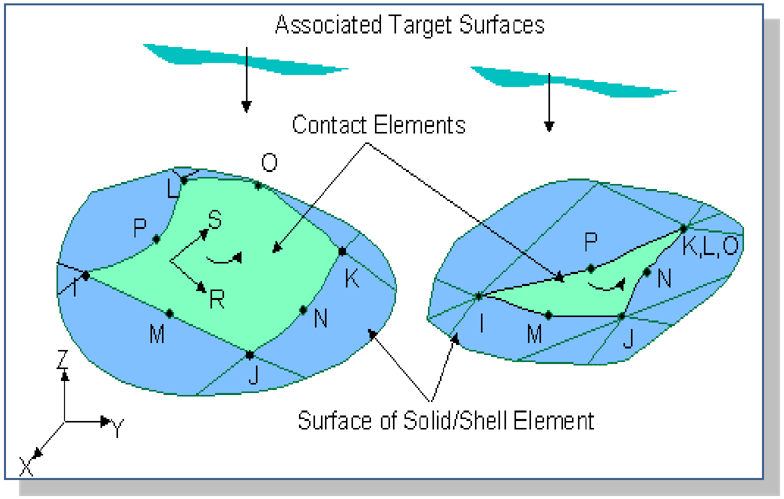
where:

c, υ = decay coefficient and slip rate, respectively;

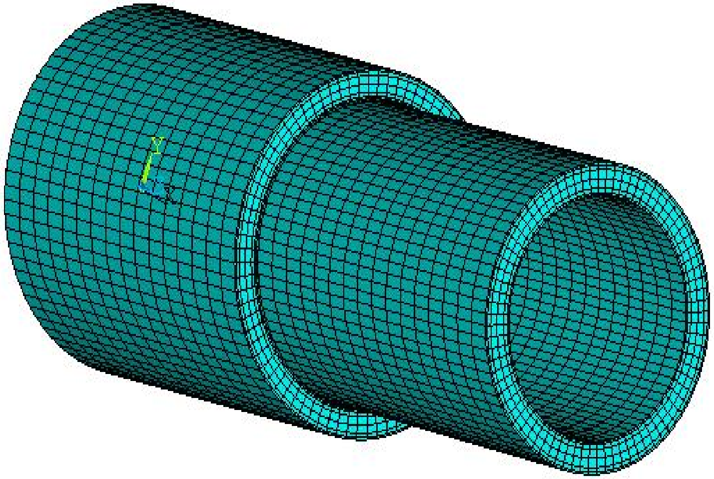
μ_d = dynamic friction coefficient;

μ_s = static friction coefficient.

Contact Model



4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER – CONT.



Model is in transient condition with the first 10 sub steps used for assembly and the next 10 sub steps used to apply internal pressure

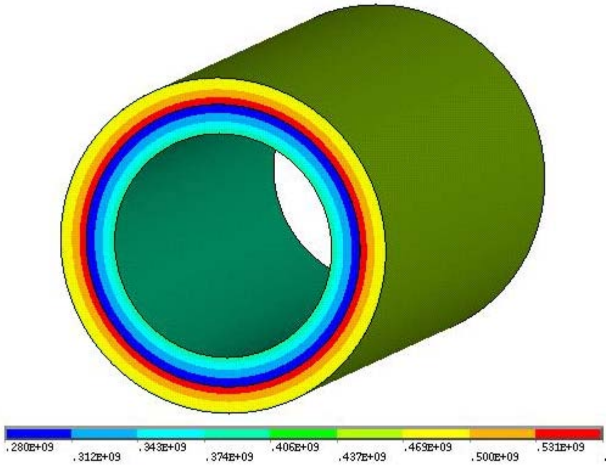
- **Two layer cylinder tapered in dimension so that one can slide into another.**
- **Dimension- inner cylinder to outer cylinder in the model is 0.06-0.09 m**
- **Element Type- Solid45, Conta174,Targe170**
- **Total elements= 17920,**

ET1= 15360, ET2&3=1280

Total nodes = 19040

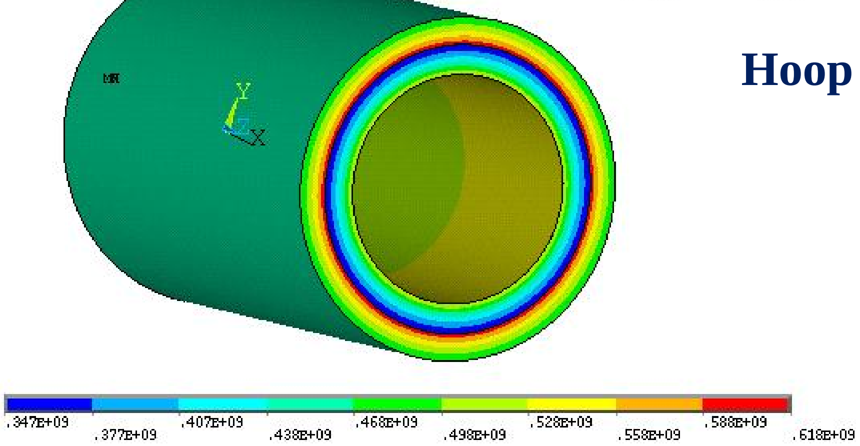
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Modeling Results - Stresses

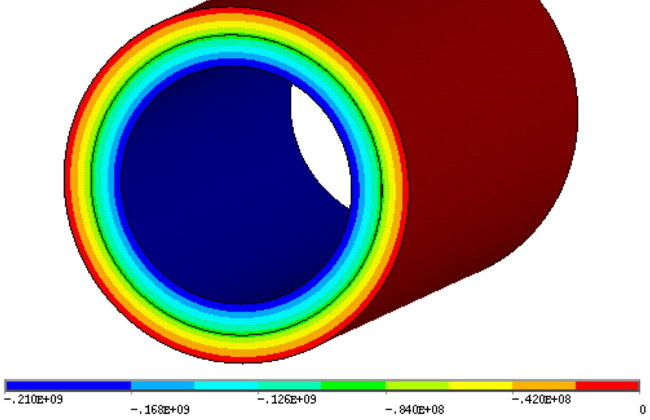


Hoop Stress

von Mises Stress



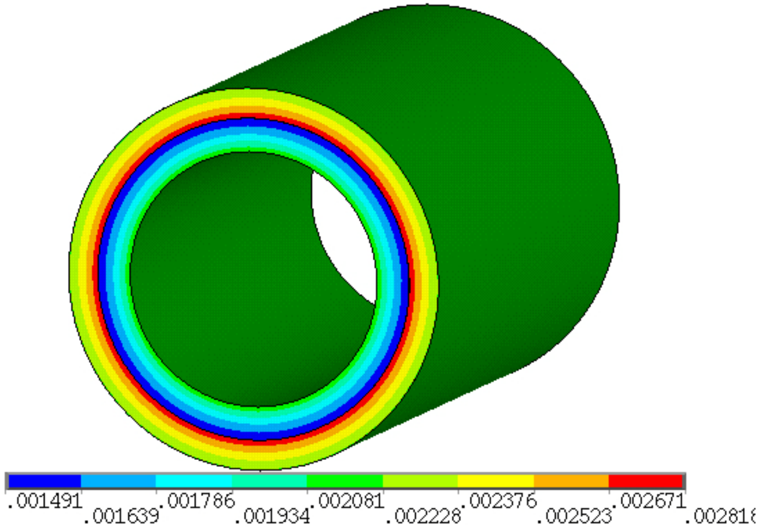
Radial Stress



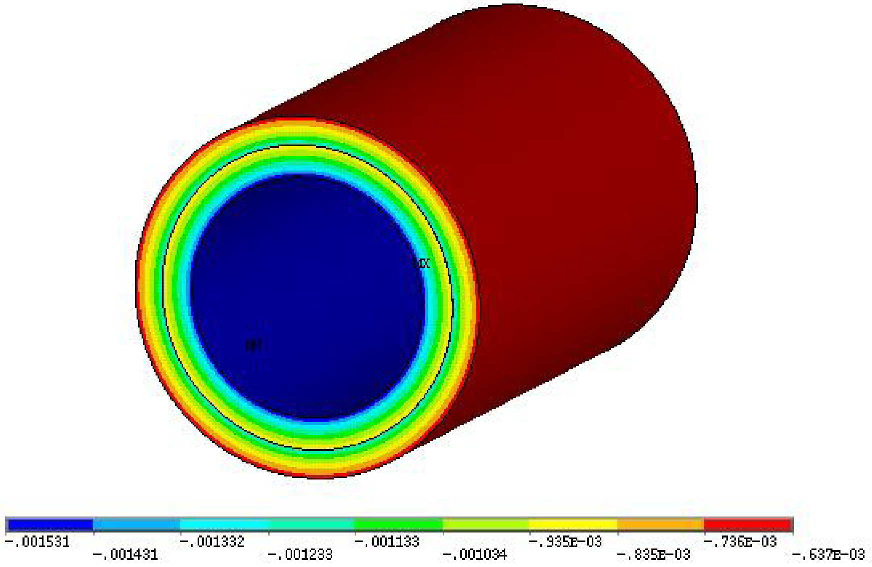
Isotropic, elastic deformation with uniform pressure of 2.1e8 Pa

4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

Modeling Results - Strains



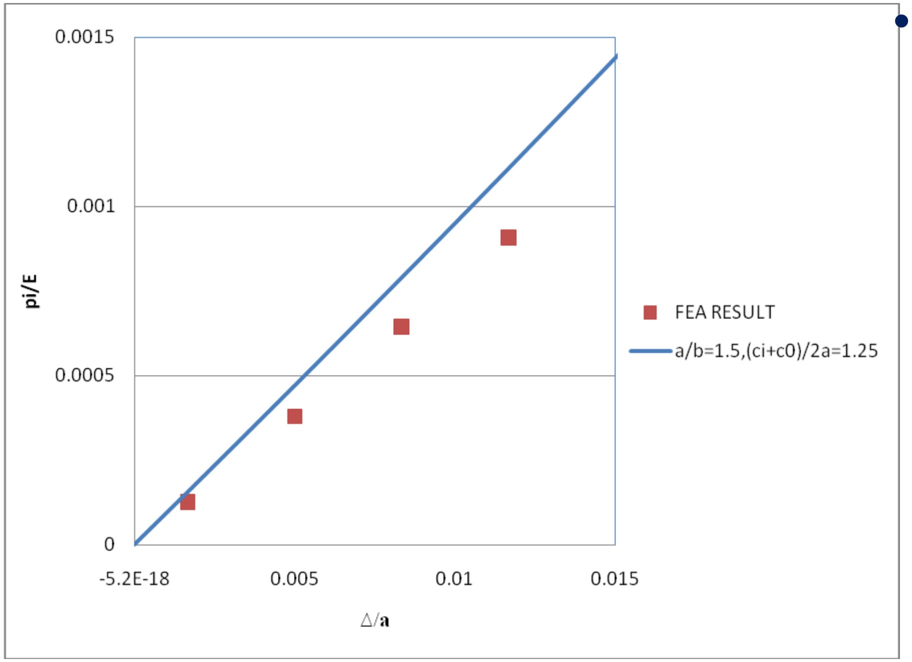
Hoop Strain



Radial Strain

4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

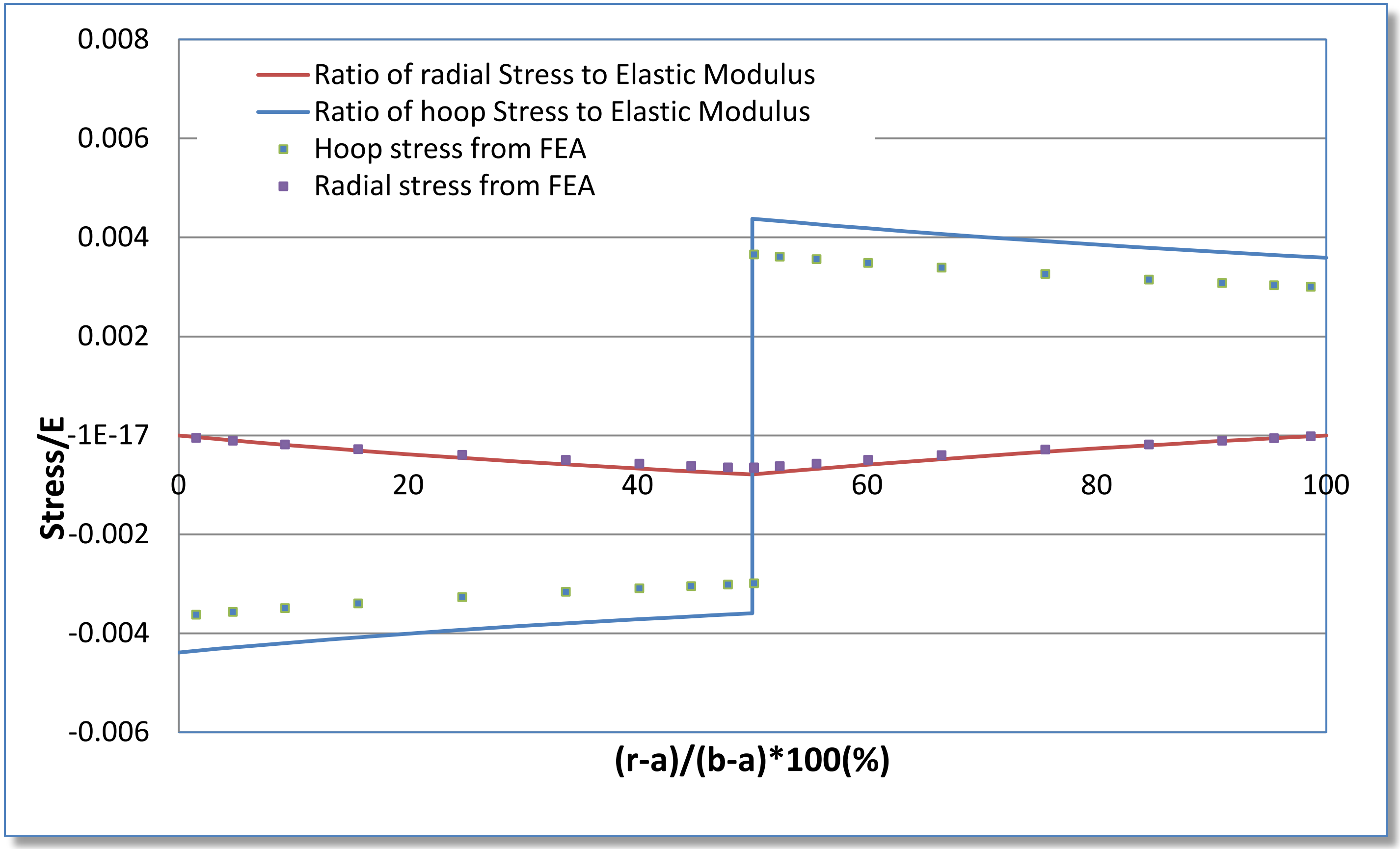
Relationship between pi/E and Δ/a



- **Relationship for dimensionless radii Δ/a & dimensionless pre-pressure after assembly pi/E.**
- **To generate more residual stress, more overlapping between the interface- more pressure**

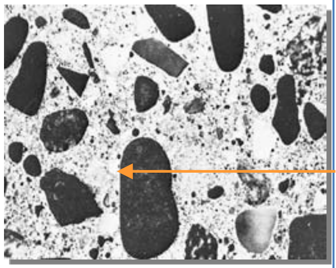
4. STRESS ANALYSIS OF A DOUBLE-LAYER PRESSURIZED THICK-WALLED CYLINDER - CONT.

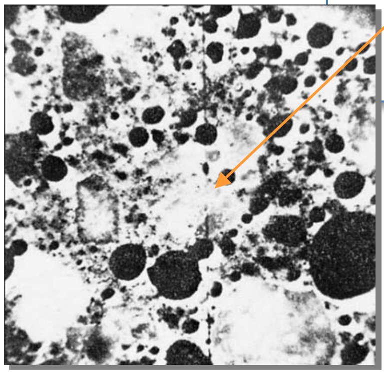
Radial and hoop stress distributions by prestressed assembly

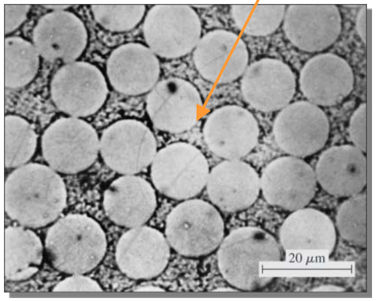


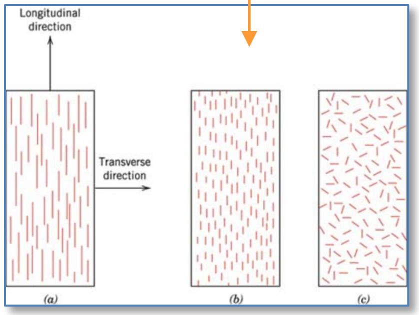
5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER

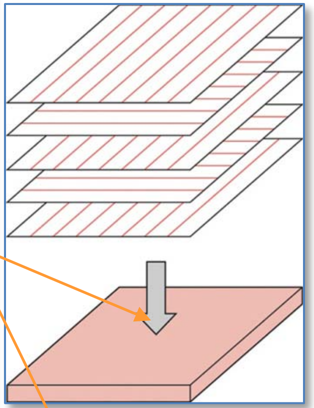
- Composites
  - Particle-reinforced
    - Large-particle
    - Dispersion-strengthened
  - Fiber-reinforced
    - Continuous (aligned)
    - Discontinuous (short)
      - Aligned
      - Randomly oriented
  - Structural
    - Laminates
    - Sandwich panels

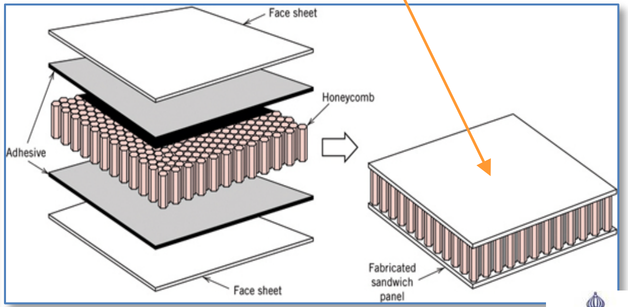




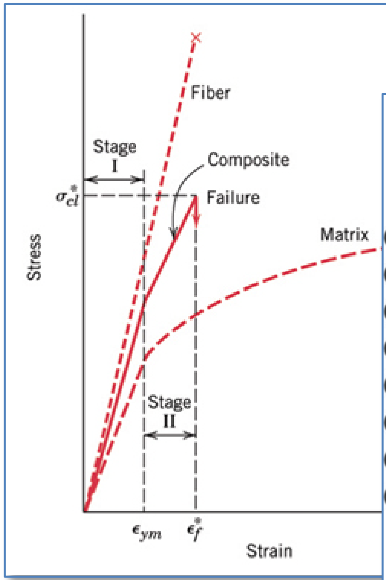


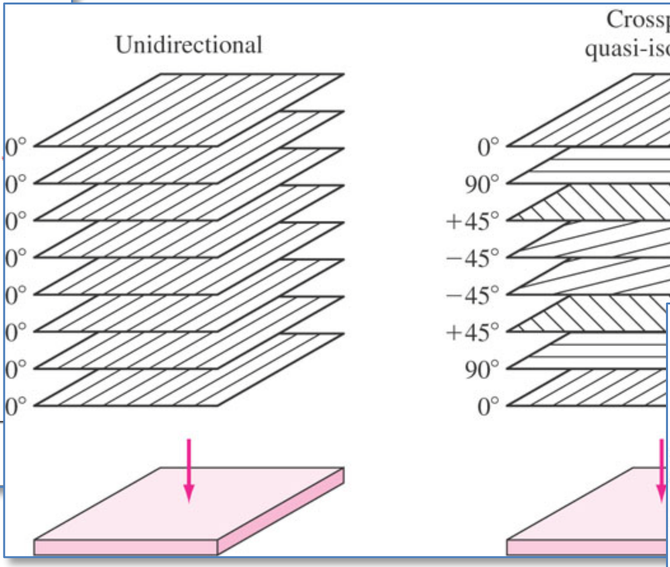


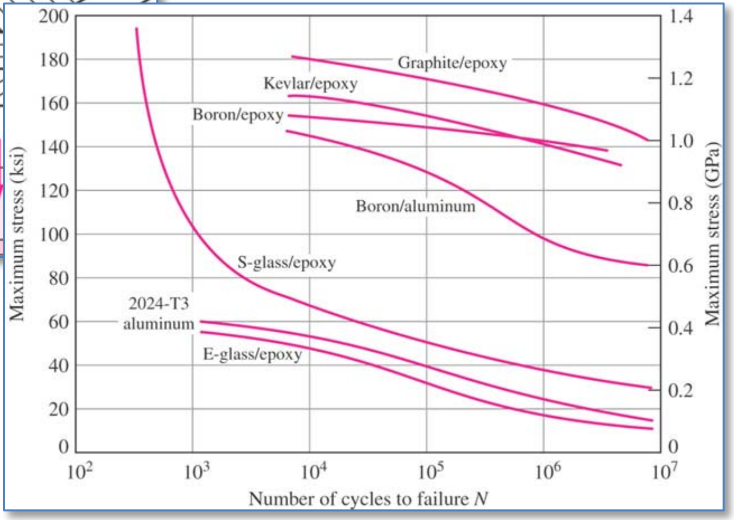




5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.







Design in Laminated Composites

5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.

The elastic material property matrix $[D]_j$ for the layer j

$$[D]_j = \begin{bmatrix} BE_{x,j} & B\nu_{xy,j}E_{x,j} & 0 & 0 & 0 & 0 \\ B\nu_{xy,j}E_{x,j} & BE_{y,j} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{xy,j} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{G_{yz,j}}{f} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{G_{xz,j}}{f} \end{bmatrix}$$

where:

$$B = \frac{E_{y,j}}{E_{y,j} - (\nu_{xy,j})^2 E_{x,j}};$$

$E_{x,j}$ = Young's modulus in layer x–direction of layer j;

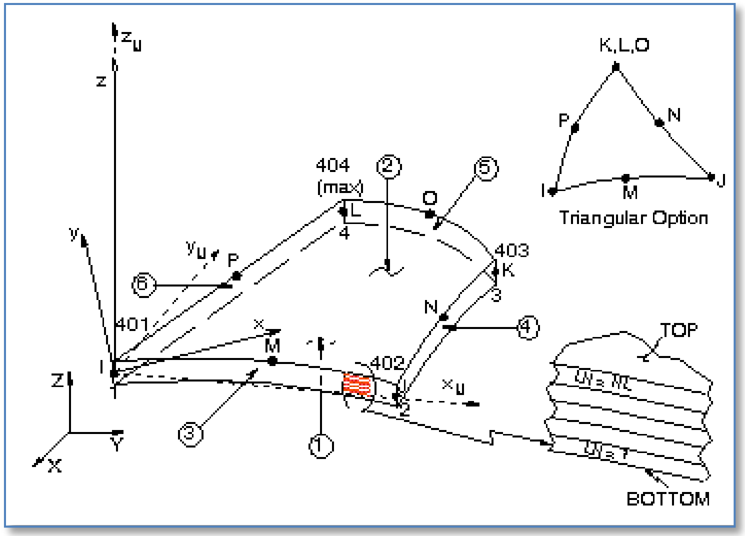
$\nu_{xy,j}$, $G_{xy,j}$ = Poisson's ratio and shear modulus in layer x–y plane of layer j, respectively;

$$f = \left\{ \begin{matrix} 1.2 \\ 1.0 + 0.2\dfrac{A}{25t^2} \end{matrix} \right\}, \text{ whichever is greater;}$$

A = element area (in s–t plane);

Governing Equations

Nonlinear Finite Strain Shell



5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.

Governing Equations

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}=0,\quad \frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}=0$$

Rewriting these in incremental form,

$$\Delta\sigma_{xz}=-\Delta z\left(\frac{\Delta\sigma_x}{\Delta x}+\frac{\Delta\sigma_{xy}}{\Delta y}\right),\quad \Delta\sigma_{yz}=-\Delta z\left(\frac{\Delta\sigma_{yx}}{\Delta x}+\frac{\Delta\sigma_y}{\Delta y}\right)$$

Setting these equations in terms of layer j,

$$\Delta\sigma_{xz,j}=-t_j\left(\frac{\Delta\sigma_{x,j}}{\Delta x}+\frac{\Delta\sigma_{xy,j}}{\Delta y}\right),\quad \Delta\sigma_{yz,j}=-t_j\left(\frac{\Delta\sigma_{yx,j}}{\Delta x}+\frac{\Delta\sigma_{y,j}}{\Delta y}\right)$$

where:

$$\Delta\sigma_{x,j}=\left(\sigma^2_{x,j}+\sigma^3_{x,j}-\sigma^1_{x,j}-\sigma^4_{x,j}\right)/2.0$$

$$\Delta\sigma_{xy,j}=\left(\sigma^3_{xy,j}+\sigma^4_{xy,j}-\sigma^1_{xy,j}-\sigma^2_{xy,j}\right)/2.0$$

$$\Delta\sigma_{yx,j}=\left(\sigma^2_{xy,j}+\sigma^3_{xy,j}-\sigma^1_{xy,j}-\sigma^4_{xy,j}\right)/2.0$$

$$\Delta\sigma_{y,j}=\left(\sigma^3_{y,j}+\sigma^4_{y,j}-\sigma^1_{y,j}-\sigma^2_{y,j}\right)/2.0$$

$\sigma^3_{x,j}$ = stress in element x direction in layer j at integration point 3.

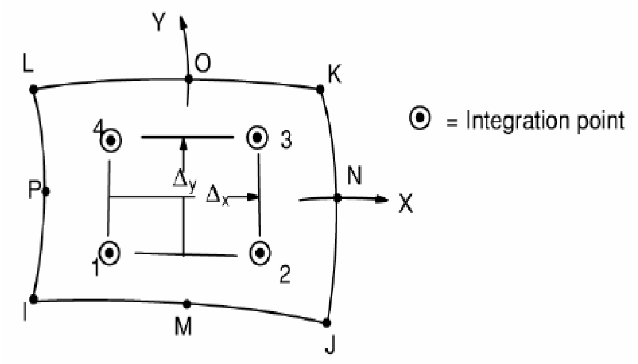


Fig.3 Integration point locations.

Δx and Δy are shown in Fig.3. Thus, the interlaminar shear stress is:

$$\tau^k_x=\sum_{j=1}^{k}\Delta\sigma_{xz,j}-S_x\sum_{j=1}^{k}t_j,\quad \tau^k_y=\sum_{j=1}^{k}\Delta\sigma_{yz,j}-S_y\sum_{j=1}^{k}t_j$$

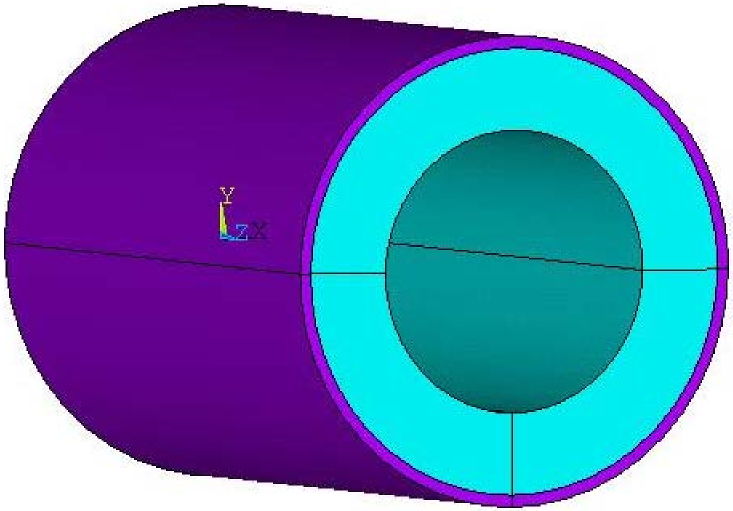
where:

τ^k_x = interlaminar shear stress between layers k and k+1;

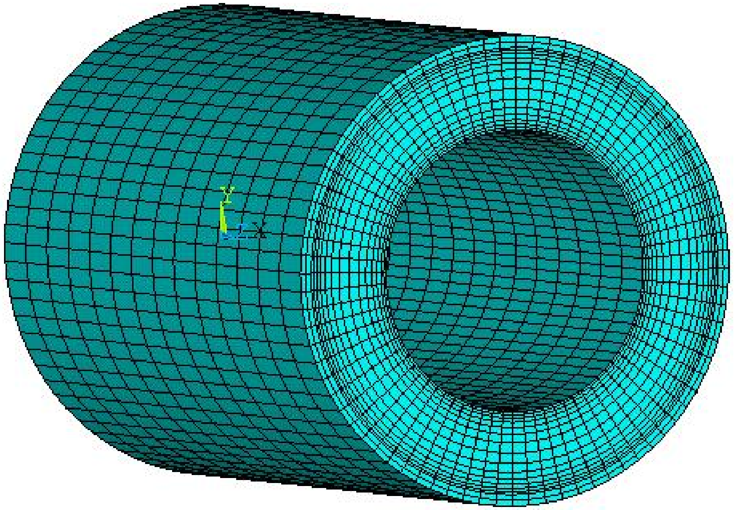
$S_x=\sum_{j=1}^{N_j}\Delta\sigma_{xz,j}\Big/t$ (= correction term);

t = total thickness.

5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER



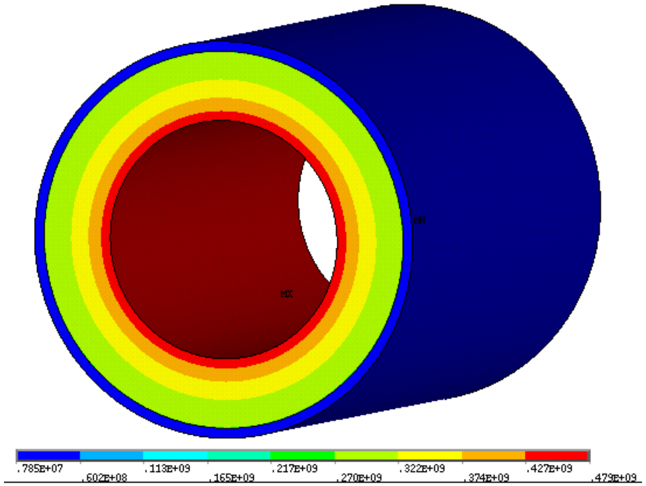
Solid Model



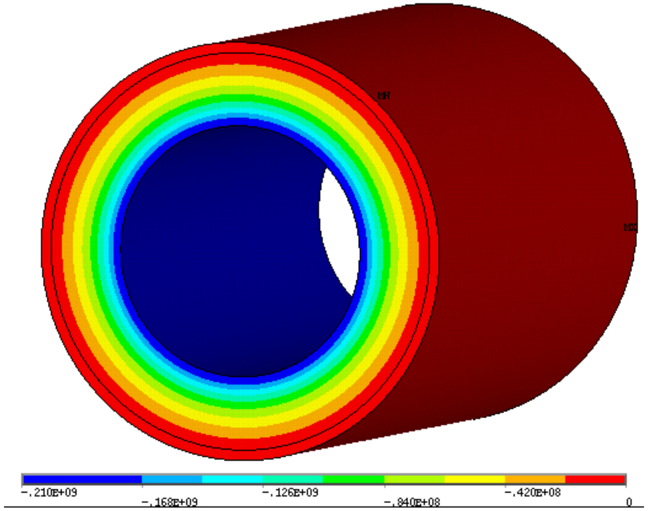
Meshed Model

Different orientations were selected, such as (0/90/0/90), (0/90/45/135), (0/90/30/120/60/150)

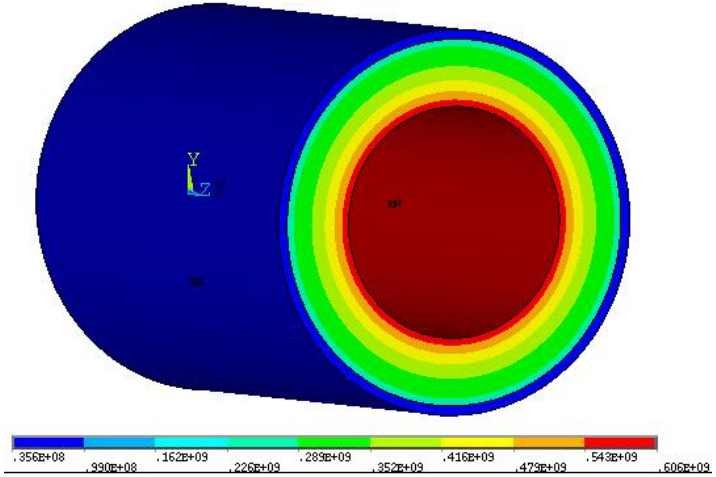
5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.



Von Mises Stress



Hoop Stress

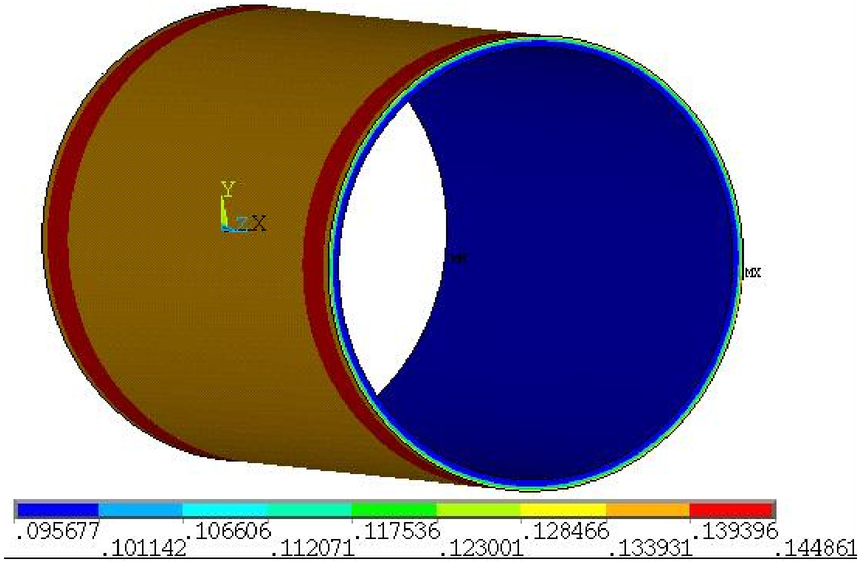


Radial Stress

Isotropic, elastic deformation with uniform pressure of 2.1e8 Pa

5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.

FAILURE CRITERIA



- Failure criteria are curve fits of experimental data that attempt to predict failure under multi-axial stress.
- Failure criteria is defined as $I_f = \frac{stress}{strength}$
- Failure is predicted when $I_f \geq 1$
- Considering failure for maximum stress criterion.

5. STRESS ANALYSIS OF A COMPOSITE-WRAPPED PRESSURIZED THICK-WALLED CYLINDER - CONT.

COMPARISON

SINGLE LAYER THICK WALL CYLINDER	**von Mises Stress** **0.671e9**
DOUBLE LAYER THICK WALL CYLINDER	**0. 618e9**
COMPOSITE WRAPPED THICK WALL CYLINDER	**0. 606e9**

Best orientation for this model was (0-90-45-135)

Acknowledgement

This work was inspirited by the DoD projects in METLAB at South Dakota State University and supported by the Department of Mechanical Engineering at SDSU. Calculation data contributed by Manjunath Gurumallappa and Sudhir Puttagunta is gratefully acknowledged.

Questions ?

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